Challenging PROBLEMS

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1. If ω is a root other than 1 of the equation $\omega^7 = 1$ prove that other roots are $\omega^2, \omega^3, \omega^4, \omega^5, \omega^6$.
If $\alpha = \omega + \omega^6$, $\beta = \omega^2 + \omega^5$, $\gamma = \omega^3 + \omega^4$, prove that the equation with roots α, β, γ is $z^3 + z^2 - 2z - 1 = 0$. Hence or otherwise, find the values of

(i) $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$

(ii) $\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}$

2. By considering the product $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$, find the necessary and sufficient condition that two roots of the equation $x^3 + px^2 + qx + r = 0$ $(r \neq 0)$ should be equal in magnitude but opposite in sign.

3. Solve the simultaneous equations
$x + y + z = 2;\ x^2 + y^2 + z^2 = 30;\ x^3 + y^3 + z^3 = 16.$

4. $ABCD$ is a plane quadrilateral whose sides CD, BA intersect at O. If P, Q are the mid-points of the diagonals AC, BD, prove that the area of ΔOPQ is one-quarter of the area of the quadrilateral $ABCD$.

5. ABC is a triangle and points L, M, N are taken on BC, CA, AB respectively such that

$$\frac{BL}{LC} = \lambda,\ \frac{CM}{MA} = \mu,\ \frac{AN}{NB} = \nu$$

Prove that $\Delta LMN = \dfrac{(1+\lambda\mu\nu)\Delta ABC}{(1+\lambda)(1+\mu)(1+\nu)}$ and deduce that if LMN is a straight line, $\lambda\mu\nu = -1$.

6. **(a)** Using vector product, solve the following three linear simultaneous equations.

$$3x + 5y + 8z = -1$$
$$2x + 3y + 4z = 3$$
$$4x + 2y + 5z = -2$$

(b) Using the method used in 6(a) find a necessary and sufficient condition for the existence of a non-trivial solution to the set of three homogeneous linear equations.

$$a_1x + b_1y + c_1z = 0,$$
$$a_2x + b_2y + c_2z = 0,$$
$$a_3x + b_3y + c_3z = 0.$$

7. If a and b are positive integers, prove that the probability that $\frac{1}{2}(a^2 + b^2)$ is a positive integer is 9/25.

8. If p is a prime and r is any integer less than $p - 1$, prove that the sum of the products of the numbers $1, 2, 3, \ldots, p - 1$ taken r together is divisible by p.

9. **(a)** Show that if m is prime to n, the equations $x^m = 1$ and $x^n = 1$, have no root common except 1.
(b) If $n = pqr$, where p, q, r are primes, show that the roots of $x^n = 1$ are the n terms of the product

$$(1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1})(1 + \beta + \beta^2 + \ldots + \beta^{q-1})(1 + \gamma + \gamma^2 + \ldots + \gamma^{r-1})$$

where α is a root of $x^p = 1$, β of $x^q = 1$, γ of $x^r = 1$.

10. **(a)** Prove that the derivative of

$$f(x) = e^x(x^2 - 6x + 12) - (x^2 + 6x + 12)$$

is never negative for any real value of x.
(b) $a_0, a_1, a_2, \ldots, a_{2n}$ are given constants and

$$P_r(x) = a_0x^r + ra_1x^{r-1} + \frac{1}{2}r(r-1)a_2x^{r-2} + \ldots + a_r$$

where $0 \le r \le 2n$. Prove that $P'_r(x) = r\,P_{r-1}(x)$ and that $\sum_{r=0}^{2n} (-1)^r\ {}^{2n}C_r\ P_r(x)\,P_{2n-r}(x)$ is a constant.

SOLUTIONS

1. $x^7 = 1,\ x = \cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7},\ n = 0, 1, 2 \ldots, 6$

$\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$, $\omega^2 = \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}$

$\omega^3 = \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7}$, $\omega^4 = \cos\frac{6\pi}{7} - i\sin\frac{6\pi}{7}$

$\omega^5 = \cos\frac{4\pi}{7} - i\sin\frac{4\pi}{7}$, $\omega^6 = \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}$

$\alpha = 2\cos\frac{2\pi}{7},\ \beta = 2\cos\frac{4\pi}{7},\ \gamma = 2\cos\frac{6\pi}{7}$. Note $\omega^7 = 1$,

$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$
$\alpha + \beta + \gamma = \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = -1$
$\alpha\beta + \beta\gamma + \alpha\gamma = (\omega + \omega^6)(\omega^2 + \omega^5) + (\omega + \omega^6)(\omega^3 + \omega^4) + (\omega^2 + \omega^5)(\omega^3 + \omega^4) = -2$
$\alpha\beta\gamma = (\omega + \omega^6)(\omega^2 + \omega^5)(\omega^3 + \omega^4) = 1$. Thus, α, β, γ are the roots of the equation

$z^3 + z^2 - 2z - 1 = 0$

Sum of the roots $= \alpha + \beta + \gamma$

$$= 2\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right) = -1$$

$$\therefore \quad \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$$

$$\cos\frac{2\pi}{7}\ \cos\frac{4\pi}{7}\ \cos\frac{6\pi}{7} = \frac{1}{8}.$$

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2. $\alpha + \beta + \gamma = -p$

$\beta + \gamma = -p - \alpha$ or, $y = -p - x$

$\therefore \quad x = -p - y.$

$$(-p-y)^3 + p(-p-y)^2 + q(-p-y) + r = 0$$

$$-(p^3 + 3p^2y + 3py^2 + y^3) + p(p^2 + y^2 + 2py) - qp - qy + r = 0$$

or, $y^3 + y^2(2p + 3p^2) + y(q - 2p^2) + r - pq = 0$

has roots $(\beta + \gamma)$, $(\gamma + \alpha)$ and $(\alpha + \beta)$. If the original equation has two roots equal but opposite in sign then

$$(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta) = 0.$$

$\therefore \quad r = pq.$

Sufficient : Let $r = pq$.

$$\Rightarrow \quad \alpha\beta\gamma = (\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$\alpha\beta\gamma = \alpha^2(\beta + \gamma) + \beta^2(\alpha + \gamma) + \gamma^2(\alpha + \beta) + 3\alpha\beta\gamma$$

or, $\alpha^2(\beta + \gamma) + \beta^2(\alpha + \gamma) + \gamma^2(\alpha + \beta) + 2\alpha\beta\gamma = 0$

$\therefore \quad (\alpha + \beta)(\beta + \gamma)(\alpha + \gamma) = 0$. Hence the result.

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3. Let x, y, z be the roots of the equation

$$t^3 - at^2 + bt - c = 0$$

Thus, $a = \sum x = 2$

$$b = \sum xy = \frac{1}{2}\left\{\left(\sum x\right)^2 - \sum (x)^2\right\} = -13$$

To find c, we use the identity

$$x^3 + y^3 + z^3 - 3abc = (x + y + z)(x^2 + y^2 + z^2 - bc - ca - ab)$$

$\therefore \quad c = 10.$

Thus x, y, z are the roots of the equation

$$t^3 - 2t^2 - 13t - 10 = 0.$$

By observation, $t = -1$ is a root.

$\therefore \quad (t + 1)(t^2 - 3t - 10) = (t + 1)(t + 2)(t - 5) = 0.$

$\therefore \quad x = -1, y = -2, z = 5.$

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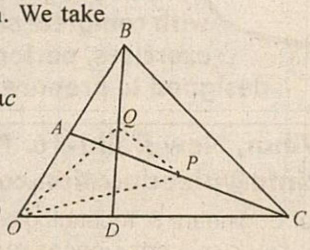
4. Take O as the origin. We take

$\overrightarrow{OA} = a$, $\overrightarrow{OC} = c$,

$\overrightarrow{OB} = b = \lambda a$, $\overrightarrow{OD} = \mu c$

$\overrightarrow{OP} = p = \frac{1}{2}(a + b)$

$\overrightarrow{OQ} = q = \frac{1}{2}(\lambda a + \mu c)$



The vector area of $OPQ = \frac{1}{2}(p \times q)$

$$= \frac{1}{8}(a + c) \times (\lambda a + \mu c) = \frac{1}{8}(\mu - \lambda)(a \times c)$$

But since $ABCD$ is a plane quadrilateral, its area is the magnitude of the sum of vector areas of ΔABC and ΔBDC. Thus,

$$ABCD = \frac{1}{2}[d \times b + b \times a + a \times d] + \frac{1}{2}[d \times c + c \times b + b \times d]$$

$$= \frac{1}{2}[b \times a + a \times d + d \times c + c \times b]$$

$$= \frac{1}{2}[a \times (d - b) - c \times (d - b)]$$

$$= \frac{1}{2}[(a - c) \times (d - b)] = \frac{1}{2}(a - c) \times (\mu c - \lambda a)$$

$$= \frac{1}{2}(\mu - \lambda)(a \times c).$$ Hence the result.

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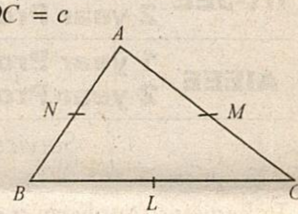
5. Take O as the origin.

$\overrightarrow{OA} = a$, $\overrightarrow{OB} = b$, $\overrightarrow{OC} = c$

Now, $\overrightarrow{OL} = \dfrac{b + \lambda a}{1 + \lambda}$,

$\overrightarrow{OM} = \dfrac{c + \mu a}{1 + \mu}$,

$\overrightarrow{ON} = \dfrac{a + \nu b}{1 + \nu}$



Vector area of ΔLMN

$$= \frac{1}{2}\left[\frac{(b + \lambda c) \times (c + \mu a)}{(1 + \lambda)(1 + \mu)} + \frac{(c + \mu a) \times (a + \nu b)}{(1 + \mu)(1 + \nu)} + \frac{(a + \nu b) \times (b + \lambda c)}{(1 + \nu)(1 + \lambda)}\right]$$

$$= \frac{1}{2}[(b \times c) + (c \times a) + (a \times b)]\frac{1 + \lambda\mu\nu}{(1 + \lambda)(1 + \mu)(1 + \nu)}.$$

If LML is a straight line then area of $\Delta LMN = 0$.

$\therefore \quad 1 + \lambda\mu\nu = 0.$

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6. **(a)** The three equations represent three planes P_1, P_2, P_3; their solution consists in finding the point Q common to each plane (assuming such a point exists). From equations P_1 and P_2, we obtain $3P_1 + P_2$, the equation of a plane through the line of intersection of P_1, P_2 containing the origin. Thus,

$$11x + 18y + 28z = 0 \quad : \quad P_4$$

is a plane containing O and Q. Similarly, $(2P_1 - P_3)$

$$2x + 8y + 11z = 0 \quad : \quad P_5$$

is a plane containing O and Q.

Thus, $\overrightarrow{OQ}$ is perpendicular to both $11\hat{i} + 18\hat{j} + 28\hat{k}$ and $2\hat{i} + 8\hat{j} + 11\hat{k}$.

$\therefore \quad \overrightarrow{OQ} = \lambda(11\hat{i}+18\hat{j}+28\hat{k})\times(2\hat{i}+8\hat{j}+11\hat{k})$

$= -13\lambda(2\hat{i}+5\hat{j}-4\hat{k})$

and Q has co-ordinates $(-26\lambda, -65\lambda, 52\lambda)$ where λ is some scalar. Write,

$-13\lambda = \mu$, Q has co-ordinates $(2\mu, 5\mu, -4\mu)$.

Substituting in P_1, we find $\mu = 1$. Hence the solution

$$x = 2, y = 5, z = -4.$$

(b) As in 6(a),

P_4 is $(a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2)z = 0$

P_5 is $(a_1 - a_3)x + (b_1 - b_3)y + (c_1 - c_3)z = 0$

$\overrightarrow{OQ}$ is perpendicular to both

$$(a_1-a_2)\hat{i}+(b_1-b_2)\hat{j}+(c_1-c_2)\hat{k}$$

and $(a_1-a_3)\hat{i}+(b_1-b_3)\hat{j}+(c_1-c_3)\hat{k}$

$$\overrightarrow{OQ} = \lambda[\{(a_1-a_2)\hat{i}+(b_1-b_2)\hat{j}+(c_1-c_2)\hat{k}\} \times \{(a_1-a_3)\hat{i}+(b_1-b_3)\hat{j}+(c_1-c_3)\hat{k}\}]$$

$$=\lambda[\hat{i}\big((b_1-b_2)(c_1-c_3)-(c_1-c_2)(b_1-b_3)\big) + \hat{j}\big((c_1-c_2)(a_1-a_3)-(a_1-a_2)(c_1-c_3)\big) + \hat{k}\big((a_1-a_2)(b_1-b_2)-(b_1-b_2)(a_1-a_3)\big)]$$

$$=\lambda[\hat{i}\,\{(b_2c_3-c_2b_3)+(b_1c_2-c_1b_2)+(c_1b_3-b_1c_3)\} + \hat{j}\,\{(c_2a_3-a_2c_3)+(c_1a_2-a_1c_2)+(a_1c_3-c_1a_3)\} + \hat{k}\,\{(a_2b_3-b_2a_3)+(a_1b_2-b_1a_2)+(b_1a_3-a_1b_3)\}]$$

Q has co-ordinates

$(\lambda(b_2c_3 - c_2b_3) + \lambda(b_1c_2 - c_1b_2) + \lambda(c_1b_3 - b_1c_3),$

$\lambda(c_2a_3 - a_2c_3) + \lambda(c_1a_2 - a_1c_2) + \lambda(a_1c_3 - c_1a_3),$

$\lambda(a_2b_3 - b_2a_3) + \lambda(a_1b_2 - b_1a_2) + \lambda(b_1a_3 - a_1b_3))$

Q lies on P_1

$$\therefore \quad a_1(b_2c_3-c_2b_3)\lambda+a_1(b_1c_2-c_1b_2)\lambda+\lambda a_1(c_1b_3-b_1c_3) + b_1(c_2a_3-a_2c_3)\lambda+b_1(c_1a_2-a_1c_2)\lambda+\lambda b_1(a_1c_3-c_1a_3) + a_1(a_2b_3-b_2a_3)\lambda+c_1(a_1b_2-b_1a_2)\lambda+\lambda c_1(b_1a_3-a_1b_3) = 0 \quad \text{... (i)}$$

We denote $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta$

$A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3$ are co-factors of $a_y, b_y, c_y, y = 1, 2, 3$. Thus,

$\lambda\Delta + \lambda(a_1A_3 + b_1B_3 + c_1C_3) + \lambda(a_1A_2 + b_1B_2 + c_1C_2) = 0$

or, $\lambda\Delta = 0$, λ can not be zero. $\quad \therefore \ \Delta = 0$.

It is left as an exercise for the students to prove that it is sufficient.

7. We take 1, 2, 3, ... 5 N. Now we divide this set in the following manner.

$\{5, 10, ..., 5N\} = A(0)$ $\quad \{1, 6, ... 5N - 4\} = A(1)$

$\{2, 7, ..., 5N - 3\} = A(2)$ $\quad \{3, 8, ... 5N - 2\} = A(3)$

$\{4, 9, ... 5N - 1\} = A(4)$

Sample space $= {}^{5n}C_2$

(i) If $a, b \in A(0)$, then $\dfrac{a^2+b^2}{5}$ is a +ve integer.

Number of favourable cases $= {}^nC_2$

(ii) If $a \in A(1), b \in A(2)$, then $\dfrac{a^2+b^2}{5}$ is a +ve integer.

$\therefore$ Number of favourable cases $= n^2$.

(iii) If $a \in A(1)$, $b \in A(3)$, then $\dfrac{a^2+b^2}{5}$ is an integer.

$\therefore$ Number of favourable cases $= n^2$.

(iv) If $a \in A(2)$, $b \in A(4)$, then $\dfrac{a^2+b^2}{5}$ is an integer.

$\therefore$ Number of favourable cases $= n^2$.

(v) If $a \in A(3)$, $b \in A(4)$, then $\dfrac{a^2+b^2}{5}$ is an integer.

$\therefore$ Number of favourable cases $= n^2$

Total favourable cases $= {}^nC_2 + 4n^2$

$$\text{Probability} = \lim_{n\to\infty} P_n = \lim_{n\to\infty} \frac{{}^nC_2+4n^2}{\frac{5n(5n-1)}{2}} = \frac{9}{25}.$$

8. Let $f(x) = (x + 1)(x + 2) ... (x + p - 1)$; we denote α_r the sum of the products of 1, 2, 3, ..., $p - 1$ taken r together.

$f(x) = x^{p-1} + \alpha_1 x^{p-2} + \alpha_2 x^{p-3} + ... + \alpha_{p-1} \quad$ ... (i)

Now we have, $(x + p) f(x) = (x + 1) f(x + 1)$ ... (ii)

i.e. $(x + p)(x^{p-1} + \alpha_1 x^{p-2} + \alpha_2 x^{p-3} + ... + \alpha_{p-1})$

$= (x + 1)^p + \alpha_1(x + 1)^{p-1} + \alpha_2(x + 1)^{p-2} + ... + \alpha_{p-1}(x + 1) \quad$ ... (iii)

Equating the coefficient of $x^{p-2}, x^{p-3}, ..., x$, we have

$p\alpha_1 = {}^pC_2 + {}^{p-1}C_1\,\alpha_1$

$p\alpha_2 = {}^pC_3 + {}^{p-1}C_2\,\alpha_1 + {}^{p-1}C_1\alpha_2$

....

....

$p\alpha_{p-2} = {}^pC_{p-1} + {}^{p-1}C_{p-2}\,\alpha_1 + ... + {}^2C_1\,\alpha_{p-2}$

Now ${}^pC_r$ is divisible by p, $r < p$; and ${}^{p-1}C_r, {}^{p-2}C_r$ etc. are all prime to p.

If follows in succession that

$\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{p-2}$ are all divisible by p.

9. (a) Let α be a common root, then

$\alpha^{pm} = 1$ and $\alpha^{qn} = 1$, where p, q are any positive integers.

Therefore $\alpha^{pm-qn} = 1$, since m is prime to n, p and q can be found so that $pm - nq = \pm 1 \Rightarrow \alpha = 1$.

(b) Take the case of three factors p, q, r; similar reason apply for all cases.

Any term of the product, for instance $\alpha^a \beta^b \gamma^c$ is a root.

$(\alpha^a)^n = (\alpha^p)^{aqr} = 1,$

and similarly, $(\beta^p)^n = 1$, and $(\gamma^c)^n = 1$.
If any two terms are equal, for instance, if

$\alpha^a \beta^b \gamma^c = \alpha^{a'} \beta^{b'} \gamma^{c'}$, then

$\beta^{b-b'} \gamma^{c-c'} = \alpha^{a'-a}$ which is impossible for $\beta^{b-b'} \gamma^{c-c'}$ is a root of $x^{qr} - 1 = 0$ and $\alpha^{a-a'}$ is a root of $x^{p-1} = 0$, since p is prime to qr, these equations have no common root except 1.

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10. (a) Let $F'(x) = f(x)$. Then

$F(x) = e^x(x^2 - 4x + 6) - (2x + 6)$ and $f(0) = 0$

$f'(x) = e^x(x^2 - 2x + 2) - 2$ and $f'(0) = 0$

$f''(x) = e^x x^2$ and is positive when $x \neq 0$.

When $x > 0$,

$f'(0) = 0, f''(x) > 0$ together gives $f'(x) > 0$;

$f(0) = 0, f'(x) > 0$ together gives $f(x) > 0$;

$F(0) = 0, F'(x) = f(x) > 0$ together gives $F(x) > 0$.

When $x < 0$

$f'(0) = 0, f''(x) > 0$ together gives $f'(x) < 0$;

$f(0) = 0, f'(x) < 0$ together gives $f(x) > 0$;

$F(0) = 0, F'(x) = f(x) > 0$ together gives $F(x) < 0$.

Thus $f(x)$ is never negative when x is real and $F(x)$ has the sign of x when $x \neq 0$.

(b) $P_r'(x) = ra_0 x^{r-1} + ra_1(r-1)x^{r-2}$

$$+ \frac{1}{2} r(r-1)(r-2)a_2 x^{r-3} + ... + {}^rC_{r-1}\, a_{r-1}$$

$$= r\left[a_0 x^{r-1} + a_1(r-1)x^{r-2} + \frac{1}{2}a_2(r-1)(r-2)x^{r-3} + ...+ a_{r-1}\right] = r\, P_{r-1}(x) \qquad ...\text{(i)}$$

Note $P_{-1} \equiv 0$

Differentiating,

$$F'(x) = \sum_{r=0}^{2n} (-1)^r\, {}^{2n}C_r\, r\, P_{r-1}(x)\, P_{2n-r}(x)$$

$$+ \sum_{r=0}^{2n} (-1)^r\, {}^{2n}C_r\, P_r(x)(2n-r)P_{2n-r-1}(x) \qquad ...\text{(ii)}$$

$$= \sum_{r=0}^{2n} (-1)^r \left\{\frac{2n!}{(2n-r)!\,(r-1)!} P_{r-1}P_{2n-2} + \frac{2n!}{(2n-r-1)!\,r!} P_r\, P_{2n-r-1}\right\}$$

$$= \sum_{r=0}^{2n} (-1)^r \{a_r + b_r\} \text{ where}$$

$$a_r = \frac{2n!}{(2n-r)!\,(r-1)!} P_{r-1}\, P_{2n-r} \qquad ...\text{(A)}$$

$$\text{and } b_r = \frac{2n!}{(2n-r-1)!\,r!} P_r\, P_{2n-r-1} \qquad ...\text{(B)}$$

$$\therefore\ b_r = a_{r+1} \qquad ...\text{(iii)}$$

We put $r = 0, 1, 2, ..., 2n$ and using the relation (ii), we get $F'(x) = 0$. Hence the result. ■

TIPS *for solving* MCQ'S

- The multiple choice question, consists of two parts:
  1. The stem - the statement or question.
  2. The choices - also known as the distracters. There are usually 3 to 5 options, that will complete the stem statement or question.

You are to select the correct choice, the option that completes the thought expressed in the stem. There is a 20% chance that you will guess the correct choice if there are 5 choices listed. Although multiple choice questions are most often used to test your memory of details, facts, and relationships, they are also used to test your comprehension and your ability to solve problems. Reasoning ability is a very important skill for doing well on multiple choice tests.

Read the stem as if it were an independent, free-standing statement. Anticipate the phrase that would complete the thought expressed, then compare each answer choice to your anticipated answer. It is important to read each choice, even if the first choice matches the answer you expected, because there may be a better answer listed.

- Another evaluation technique is to read the stem together with each answer choice as if it were a true-false statement. If the answer makes the statement a false one, cross it out. Check all the choices that complete the stem as a true statement. Try to suspend judgment about the choices you think are true until you have read all the choices.
- Beware of words like not, but, except. Mark these words because they specify the direction and limits of the answer.
- Also watch out for words like always, never, and only. These must be interpreted as meaning all of the time, not just 99% of the time. These choices are frequently incorrect because there are few statements that have no exceptions (but there are a few).
- If there are two or more options that could be the correct answer, compare them to each other to determine the differences between them, and then relate these differences with the stem to deduce which of the choices is the better one. (Hint: select the option that gives the most complete information.)
- If there is an encompassing answer choice, for example "all of the above," and you are able to determine that there are at least two correct choices, select the encompassing choice.
- Use hints from questions you know to answer questions you do not.
- If you do not find an answer, try to relate each answer to the stem to evaluate which one logically completes the thought.
- Make educated guesses—eliminate options any way you can.

INVERSE TRIGONOMETRIC FUNCTIONS

(Concept & Analysis)

By : **Prof. S.S. Dahiya**, Director Academics,
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$$\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}\Leftrightarrow \sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$$

Inverse Trigonometric function	Domain	Range
$y=f(x)=\sin^{-1}(x)$	$-1\le x\le 1$	$-\frac{\pi}{2}\le y\le\frac{\pi}{2}$
$y=g(x)=\cos^{-1}(x)$	$-1\le x\le 1$	$0\le y\le\pi$
$y=\phi(x)=\tan^{-1}(x)$	$x\in$ Real	$-\frac{\pi}{2}<y<\frac{\pi}{2}$
$y=\Psi(x)=\cot^{-1}(x)$	$x\in$ Real	$0<y<\pi$
$y=G(x)=\sec^{-1}(x)$	$x\le-1$ or $x\ge1$	$0\le y\le\pi,\ y\ne\frac{\pi}{2}$
$y=F(x)=\text{cosec}^{-1}(x)$	$x\le-1$ or $x\ge1$	$-\frac{\pi}{2}\le y\le\frac{\pi}{2}, y\ne0$

1. $\text{cosec}^{-1}(k)=\sin^{-1}\left(\frac{1}{k}\right)$
2. $\sec^{-1}(t)=\cos^{-1}\left(\frac{1}{t}\right)$
3. $\cot^{-1}(q)=\begin{cases}\tan^{-1}\left(\frac{1}{q}\right), \text{ if } q>0\\ \tan^{-1}\left(\frac{1}{q}\right)+\pi \text{ if } q<0\end{cases}$
4. $\sin^{-1}(x)+\cos^{-1}(x)=\frac{\pi}{2},\ -1\le x\le 1$
5. $\tan^{-1}(k)+\cot^{-1}(k)=\frac{\pi}{2}$
6. $\sec^{-1}(t)+\text{cosec}^{-1}(t)=\frac{\pi}{2}, t\le-1 \text{ or } t\ge+1$
7. $\sin^{-1}(-x)=-\sin^{-1}(x)$
8. $\tan^{-1}(-k)=-\tan^{-1}(k)$
9. $\cos^{-1}(-\lambda)=\pi-\cos^{-1}(\lambda)$

10. Determinor for $\sin^{-1}(x)$ is π (explain the rearson) If odd number of determinors are used than sign change gives answer, if even number of determinors are used then there is no sign change.

Example : $\sin^{-1}(\sin 10)=\theta$ then find θ

Because $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ whereas 10 radians $\approx$ 570 degrees 10 radians = $3\pi+30°$ approx or $10-3\pi$ lies within the required range, three determinors are used, hence $\theta=-(10-3\pi)=3\pi-10$.

11. Determinor for $\tan^{-1}(k)$ is π and there is no criteria of sign change (explain the reason)
$\tan^{-1}(\tan 10)=\alpha=$, find α

Because $-\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$, $10-3\pi$ is approx. 30°
$\therefore\ \alpha=10-3\pi$

12. Determinor for $\cos^{-1}(t)$ is 2π (explain the reason) In first step adjust complete rounds, if angle left $\in[0,\pi]$ then angle left is answer, if angle left $\in(\pi, 2\pi)$ then 2π – angle left gives answer $\cos^{-1}(\cos 10)=\theta$, find θ.
10 radians = $2\pi+210°$ approx.
$10-2\pi=210°$ approx. $\in(\pi,2\pi)$
$\therefore\ \theta=2\pi-(10-2\pi)=4\pi-10$,
$\cos^{-1}(\cos 15)=\alpha$, find α
15 radians = $2\pi+2\pi+135°$ approx.
$15-4\pi=135°$ approx $\in[0,\pi]$ hence $\theta=15-4\pi$

13. Find x if $\frac{2\pi}{3}=\tan^{-1}(x)$, because $\frac{-\pi}{2}<\tan^{-1}(x)<\frac{\pi}{2}$ there $\tan^{-1}(x)=\frac{2\pi}{3}$ is not possible, hence no value of x.

14. $\theta=\tan^{-1}(x)+\tan^{-1}(y)+\tan^{-1}(z)$ and $\tan(\theta)=0$ then find all possible values of θ

Because $\frac{-3\pi}{2}<\tan^{-1}(x)+\tan^{-1}(y)+\tan^{-1}(z)<\frac{3\pi}{2}$

$\therefore\ \theta\in\{-\pi,0,\pi\}$.

15. r, x, y, z are real numbers such that $r^2 = x^2 + y^2 + z^2$ and $\tan^{-1}\left(\frac{rx}{yz}\right)+\tan^{-1}\left(\frac{ry}{zx}\right)+\tan^{1}\left(\frac{rz}{xy}\right)=\theta$ then find all possible values of θ.

$$\tan^{-1}\left(\frac{rx}{yz}\right)+\tan^{-1}\left(\frac{ry}{zx}\right)=\tan^{-1}\left(\frac{\frac{rx}{yz}+\frac{ry}{zx}}{1-\frac{r^2}{z^2}}\right)=\tan^{-1}\left(\frac{-rz}{xy}\right)$$

$$\therefore \tan\theta=\tan\left(\tan^{-1}\left(\frac{rz}{xy}\right)+\tan^{-1}\left(\frac{-rz}{xy}\right)\right)=0$$

$\therefore \theta \in \{-\pi, \pi\}$, here $\theta \neq 0$ because r, x, y, z cannot be zero and either all of $\frac{rx}{yz}, \frac{ry}{zx}, \frac{rz}{xy}$ are positive or all are negative.

16. $\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)=$

$$\begin{cases}-\sin^{-1}(x)-\sin^{-1}(y)+\pi, & x>0, y>0, x^2+y^2>1\\ \sin^{-1}(x)+\sin^{-1}(y) & \text{if } x^2+y^2\le 1 \text{ or } xy\le 0\\ -\sin^{-1}(x)-\sin^{-1}(y)-\pi & \text{if } x<0, y<0, x^2+y^2>1\end{cases}$$

17. If $\sin^{-1}(\pm x)+\sin^{-1}(\pm y)=\pm\frac{\pi}{2}$ then $x^2+y^2=1$.

18. If $x \le 0, y \le 0$ or $x \ge 0, y \ge 0$ then $\cos^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right)=$

$$\begin{cases}\cos^{-1}(x)-\cos^{-1}(y) & \text{when } x\le y\\ \cos^{-1}(y)-\cos^{-1}(x) & \text{when } y\le x\end{cases}$$

19. If $x \le 0, y \le 0$ or $x \ge 0, y \ge 0$ then $\tan^{-1}(x)+\tan^{-1}(y)=$

$$\begin{cases}\tan^{-1}\left(\frac{x+y}{1-xy}\right)-\pi, & \text{when } x<0, y<0, xy>1\\ \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{when } 0\le xy<1\\ \tan^{-1}\left(\frac{x+y}{1-xy}\right)+\pi, & \text{when } x>0, y>0, xy>1\end{cases}$$

20. (A): $\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\begin{cases}-2\tan^{-1}(x)-\pi, & \text{when } x\le -1\\ 2\tan^{-1}(x), & \text{when } -1\le x\le 1\\ -2\tan^{-1}(x)+\pi, & \text{when } x\ge 1\end{cases}$

(B) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\begin{cases}2\tan^{-1}(x)+\pi, & x<-1\\ 2\tan^{-1}(x), & -1<x<1\\ 2\tan^{-1}(x)-\pi, & x>1\end{cases}$

(C) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\begin{cases}2\tan^{-1}(x), & x\ge 0\\ -2\tan^{-1}(x), & x\le 0\end{cases}$

(D) For domain $0 \le x < 1$,

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$
$$=2\tan^{-1}(x).$$

21.(A) $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=\begin{cases}-2\sin^{-1}(x)-\pi, & -1\le x\le -1/\sqrt{2}\\ 2\sin^{-1}(x), & -\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}\\ -2\sin^{-1}(x)+\pi, & \frac{1}{\sqrt{2}}\le x\le 1\end{cases}$

(B) $\cos^{-1}(2x^2-1)=\begin{cases}2\cos^{-1}(x), & x\ge 0\\ 2\pi-2\cos^{-1}(x), & x\le 0\end{cases}$

22. In inverse trigonometric function, start with angle θ, after simplification if angle decreases i.e. becomes $\frac{1}{2}\theta, \frac{1}{3}\theta$ etc then lies within the domain, if angle increases i.e. becomes 2θ, 3θ etc. then is liable to go out of domain and to bring it within the domain using determinor, different branches of the formulae are formed.

Example : $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$;

Put $\cos\theta = x$ $\therefore$ $\theta = \cos^{-1}(x)$, $-1 \le x \le 1$

$$\cos^{-1}(8x^4-8x^2+1)=\begin{cases}4\cos^{-1}(x), & \frac{1}{\sqrt{2}}\le x\le 1\\ 2\pi-4\cos^{-1}(x), & 0\le x\le\frac{1}{\sqrt{2}}\\ 4\cos^{-1}(x)-2\pi, & -\frac{1}{\sqrt{2}}\le x\le 0\\ 4\pi-4\cos^{-1}(x), & -1\le x\le -\frac{1}{\sqrt{2}}\end{cases}$$

23. Express $2\tan^{-1}(x)$ in terms of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

Hint use 20(B).

$$2\tan^{-1}(x)=\begin{cases}\tan^{-1}\left(\frac{2x}{1-x^2}\right)-\pi, & x<-1\\ -\frac{\pi}{2}, & x=-1\\ \tan^{-1}\left(\frac{2x}{1-x^2}\right), & -1<x<1\\ +\frac{\pi}{2}, & x=1\\ \tan^{-1}\left(\frac{2x}{1-x^2}\right)+\pi, & x>1\end{cases}$$

24. Convert $\sin^{-1}(x)-\sin^{-1}(y)=\pi/2$ in the form $y=f(x)$

$$\sin^{-1}(x)+\sin^{-1}(-y)=\frac{\pi}{2}\ \therefore\ 0\le x\le 1,$$

$$0\le -y\le 1,\ (x^2)+(-y^2)=1$$

Therefore, $0\le x\le 1,\ -1\le y\le 0$. Hence $y=-\sqrt{1-x^2}$.

25. Solve for x the equation

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{2\pi}{3}$$

Answer : $x=-\sqrt{3},\ x=-(2-\sqrt{3}),\ x=\frac{1}{\sqrt{3}},\ x=2+\sqrt{3}$.

26. In inverse trigonometric functions wrong questions have been given in various books and other sources, for example prove that

$$\frac{1}{2}\cos^{-1}\left(\frac{\cos\theta+\cos\phi}{1+\cos\theta\cos\phi}\right)=\tan^{-1}\left(\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right)$$

The question is wrong because for $\theta=\frac{2\pi}{3},\ \phi=\frac{4\pi}{3}$

Left hand side is positive whereas right hand side is negative correct version of the question is

Prove that $\frac{1}{2}\cos^{-1}\left(\frac{\cos\theta+\cos\phi}{1+\cos\theta\cos\phi}\right)=\tan^{-1}\left|\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right|$

Further $\tan^{-1}\left(\frac{a_1x-y}{a_1y+x}\right)+\tan^{-1}\left(\frac{a_2-a_1}{1+a_2a_1}\right)+\ldots$

$$+\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right)+\tan^{-1}\left(\frac{1}{a_n}\right)=\tan^{-1}\left(\frac{x}{y}\right)$$

is wrong, the correct version of the question is

$$\tan^{-1}\left(\frac{a_1x-y}{a_1y+x}\right)+\tan^{-1}\left(\frac{a_2-a_1}{1+a_2a_1}\right)+\ldots+$$

$$\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_n a_{n-1}}\right)+\tan^{-1}\left(\frac{1}{a_n}\right)=\tan^{-1}\left(\frac{x}{y}\right)+\lambda\pi$$

where $\lambda\in$ integer and $-n\le\lambda\le n$.

27. (i) Express $\sin^{-1}(3x-4x^3)$ in terms of $3\sin^{-1}(x)$, $-1\le x\le 1$

(ii) Express $\cos^{-1}(4x^3-3x)$ in terms of $3\cos^{-1}(x)$, $-1\le x\le 1$

(iii) Express $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ in terms of $3\tan^{-1}(x)$,

$x\in R-\left\{\pm\frac{1}{\sqrt{3}}\right\}$

28. (i) $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$

where $x\in[-1,1], y\in[-1,1]$

convert in inverse trigonometric function.

(ii) Express $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ in terms of $\tan^{-1}(x)$, $x\in R-\{0\}$.

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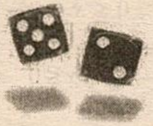


PROBABILITY THEORY
Mathematics of Chance

Probability, as the concept is most commonly understood, is a mathematical expression of the relationship between a particular outcome of an event and the total number of possible outcomes. In its commonest form, like a gambling game, probability becomes the ratio of the chances of winning — expressed in a number of ways: 50-50, a chance of 1 out of 2, 50 per cent chance, a .5 probability, and so on.

Probability is therefore concerned with the analysis of random phenomena. The outcome of a random event cannot be determined before it occurs. The actual outcome may be any one of several possible outcomes, and this is considered to be determined by chance.

Chance and Risk : The concepts of chance, fortune and luck are as old as the first dice games. Archaeologists have found evidence of games of chance in prehistoric excavations, indicating that gaming and gambling had been a major pastime for different races since the dawn of civilization.

For centuries human beings speculated about probabilities in connection with legal questions of evidence and contracts and, at times, insurance schemes. The Babylonians had several forms of maritime insurance. The Romans had annuities, i.e., exchanges of a lump sum in return for regular payments over a long time — the risk being that the person taking out the annuity would not live to collect the lump sum.

Given the penchant of the people of Greek, Egyptian, Chinese and Indian dynasties for gambling, one would expect the mathematics of chance to be one of the earliest to have developed. Surprisingly, it wasn't until the 17th century that an accurate mathematics of probability was developed by French mathematicians Pierre de Fermat and Blaise Pascal.

$$x^2-2x\left(y\frac{dy}{dx}+x\right)+y^2=0$$

$$x^2+y^2-2xy\frac{dy}{dx}-2x^2=0 \text{ or } x^2-y^2+2xy\frac{dy}{dx}=0$$

54. (d) : $9x^2+4y^2=36$

$\frac{x^2}{4}+\frac{y^2}{9}=1$ (equation of ellipse)

Remember area enclosed by ellipse is πab.

i.e. $\pi\, 2\cdot 3=6\pi$

55. (b) : $(2x-y+1)\,dx+(2y-x+1)\,dy=0$

$\frac{dy}{dx}=\frac{2x-y+1}{x-2y-1}$ put $x=x+h;\ y=y+k$

$$\therefore \frac{dy}{dx}=\frac{2x-y+2h-k+1}{x-2y+h-2k-1}$$

s.$+\,2h-k+1=0 \quad \Rightarrow h=-1$

$h-2k-1=0 \qquad k=-1 \quad \therefore \frac{dy}{dx}=\frac{2x-y}{x-2y}$

put $y=vx \quad \therefore \frac{dy}{dx}=V+X\frac{dv}{dx}$

$$\Rightarrow V+X\frac{dv}{dx}=\frac{2x-vx}{x-2vx}=\frac{2-v}{1-2v}$$

$$X\frac{dv}{dx}=\frac{2-v-2v^2}{1-2v}=\frac{2v^2-2v+2}{1-2v} \quad \therefore \frac{dx}{x}=\frac{(1-2v)dv}{2v^2-2v+2}$$

put $v^2-v+1=t \Rightarrow (2v-1)\,dv=dt$

$\therefore \frac{dx}{x}=\frac{-dt}{2t} \quad \therefore \log x=\log t^{-1/2}+\log c$

$\therefore x=t^{-1/2}c \Rightarrow x=(v^2-v+1)\cdot C$

$x^2=(v^2-v+1)=$ constt.

$$(x+1)^2\left(\frac{(y+1)^2}{(x+1)^2}-\frac{(y+1)}{x+1}+1\right)=\text{constt.}$$

$(y+1)^2-(y+1)(x+1)+(x+1)^2=c$

$y^2+x^2-xy+2y+2x-x-y+1-1+1=c$

$y^2+x^2-xy+x+y=c$

56. (d) : $y=\tan^{-1}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$

put $x^2=\cos 2\theta$ or $\theta=\frac{1}{2}\cos^{-1}x^2$

$$y=\tan^{-1}\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}$$

$$=\tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)=\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$=\tan^{-1}\tan\left(\frac{\pi}{4}-\theta\right)=\frac{\pi}{4}-\theta=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x^2$$

$$\therefore \frac{dy}{dx}=+\frac{1}{2}\frac{1}{\sqrt{1-(x^2)^2}}\,2x=\frac{+x}{\sqrt{1-x^4}}$$

57. (d) : $x=\sin t;\ y=\cos pt$

$$\frac{dx}{dt}=\cos t\ ;\ \frac{dy}{dt}=-p\sin pt\ ;\ \frac{dy}{dx}=\frac{-p\sin pt}{\cos t}$$

$$\frac{d^2y}{dx^2}=\frac{-\cos t\ p^2\cos pt(dt/dx)-p\sin pt\sin t\,(dt/dx)}{\cos^2 t}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}-\frac{xdy}{dx}+p^2y=0$$

58 (a)

59 (c) : Solving equations $x^2+y^2=5$ and $y^2=4x$ we get $x^2+4x-5=0$ *i.e.* $x=1,-5$

for $x=1;\ y^2=4;\Rightarrow y=\pm 2$ for $x=-5;\ y^2=-20$ (imaginary values) $\therefore$ points are $(1,2)\ (1,-2)$

m_1 for $x^2+y^2=5$ at $(1,2)$

$\left.\frac{dy}{dx}=-\frac{x}{y}\right|_{(1,2)}=-\frac{1}{2}$ Similarly, m_2 for $y^2=4x$ at $(1,2)$ is 1

$$\therefore \tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{-\frac{1}{2}-1}{1-\frac{1}{2}}\right|=3$$

60. (b) : Volume $=v=\frac{4}{3}\pi r^3\ ,\quad \frac{dv}{dt}=\frac{4}{3}\pi\cdot 3r^2\frac{dr}{dt}$

$\Rightarrow \frac{dv}{dt}=4\pi r^2\cdot\frac{dr}{dt}$ at $r=7$ cm

35 cc/min $=4\pi(7)^2\frac{dr}{dt} \Rightarrow \frac{dr}{dt}=\frac{35}{4\pi(7)^2}$

S.A. $=4\pi r^2$

$$\frac{ds}{dt}=8\pi r\frac{dr}{dt}=\frac{8\pi\cdot 7\cdot 35}{4\pi(7)^2}=10\ \text{cm}^2/\text{min}$$

■

Answers for SUDOKU Challenge

5	1	8	4	9	3	6	2	7
2	6	3	7	1	8	4	9	5
7	4	9	2	6	5	1	3	8
6	8	1	9	7	2	3	5	4
3	9	5	6	8	4	7	1	2
4	7	2	3	5	1	8	6	9
1	5	6	8	2	7	9	4	3
9	3	7	5	4	6	2	8	1
8	2	4	1	3	9	5	7	6

Mathematics Olympia for IIT-JEE (MAINS)

By : **Er. Akhlak Ahmad**, ABC Classes, Gorakhpur

1. Use integral calculus to find the sum of the series $\frac{1}{4!}+\frac{4!}{8!}+\frac{8!}{12!}+....\infty$.

2. Find the minimum odd value of a (> 1) such that $\int_{10}^{19}\frac{\sin x}{1+x^a}dx<\frac{1}{9}$.

3. Find the area of the region bounded by the curve $2^{|x|}|y|+2^{|x|-1}\le 1$, with in the square formed by the lines $|x|\le 1/2$, $|y|\le 1/2$.

4. Let $f(x+2y)=f(x)(f(y))^2$ for all x and y. If $f'(0)=\ln 2$, then prove that
$$f(x)+f(2x)+f(3x)+...+f(nx)=\frac{f(x)(f(nx)-1)}{f(x)-1}$$
(consider $f(x)$ is non-negative function).

5. If $H(x_0)=0$ for some $x=x_0$ and $\frac{d}{dx}H(x)>2cxH(x)\ \forall\ x\ge x_0$, where $c>0$, then prove that $H(x)$ cannot be zero for any $x>x_0$.

6. Find the domain of the function
$$f(x)=\frac{5}{\left[\frac{x-1}{2}\right]}-3^{\sin^{-1}x^2}+\frac{(7x+1)!}{\sqrt{x+1}},$$
(where $[\cdot]$ represents the greatest integer function.)

7. Integrate $\int\frac{dy}{y^2(1+y^2)^3}$.

8. A tangent to the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$ cuts the circle $x^2+y^2=4$ at points A and B, C is any point on the circle $x^2+y^2=4$ such that A, B and C are to the same side of x-axis. Find the maximum area of the ΔABC.

9. Let $f(x)=\begin{cases}x^3-x^2+10x-5, & x\le 1\\ -2x+\log_2(b^2-2), & x>1\end{cases}$

Find all possible real values of b such that $f(x)$ has the greatest value at $x=1$.

10. A curve with equation of the form $y=ax^4+bx^3+cx+d$ has zero gradient at the point (0, 1) and also touches the x-axis at the point (–1, 0). Then find the values of x for which the curve has negative gradients.

SOLUTION

1. $S=\frac{1}{4!}+\frac{4!}{8!}+\frac{8!}{12!}+...\infty$

$$=\sum_{k=0}^{\infty}\frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)}$$

$$=\sum_{k=0}^{\infty}\left(\frac{1}{6(4k+1)}-\frac{1}{2(4k+2)}+\frac{1}{2(4k+3)}-\frac{1}{6(4k+4)}\right)$$

Now consider the following definite integrals,

$$\int_0^1 x^{4k}dx=\frac{1}{4k+1};\ \int_0^1 x^{4k+1}dx=\frac{1}{4k+2}$$

$$\int_0^1 x^{4k+2}dx=\frac{1}{4k+3};\ \int_0^1 x^{4k+3}dx=\frac{1}{4k+4}$$

$$\Rightarrow S=\sum_{k=0}^{\infty}\int_0^1\left(\frac{1}{6}x^{4k}-\frac{1}{2}x^{4k+1}+\frac{1}{2}x^{4k+2}-\frac{1}{6}x^{4k+3}\right)dx$$

$$=\int_0^1\sum_{k=0}^{\infty}\left(\frac{1}{6}x^{4k}-\frac{1}{2}x^{4k+1}+\frac{1}{2}x^{4k+2}-\frac{1}{6}x^{4k+3}\right)dx$$

$$=\frac{1}{6}\int_0^1\sum_{k=0}^{\infty}x^{4k}(1-3x+3x^2-x^3)\,dx=\frac{1}{6}\int_0^1\frac{(1-x)^3}{1-x^4}dx$$

$$=\frac{1}{6}\int_0^1\frac{(1-x)^2}{(1+x)(1+x^2)}dx=\frac{1}{6}\int_0^1\left(\frac{2}{1+x}-\frac{x+1}{1+x^2}\right)dx$$

Integrating, we get

$$S=\left[\frac{1}{3}\ln(1+x)-\frac{1}{12}\ln(1+x^2)-\frac{1}{6}\tan^{-1}x\right]_0^1$$

$$=\left(\frac{1}{3}-\frac{1}{12}\right)\ln 2-\frac{1}{6}\tan^{-1}(1)=\frac{1}{4}\ln 2-\frac{\pi}{24}.$$

2. $I=\int_{10}^{19}\frac{\sin x}{1+x^a}dx<\int_{10}^{19}\frac{1}{1+x^a}dx<\int_{10}^{19}\frac{1}{1+10^a}dx$

or $1<\frac{9}{1+10^a}$ or, $1<\frac{9}{1+10^a}<\frac{1}{9}$

or, $1 + 10^a > 81 \Rightarrow a = 2, 3, 4, 5, ...$

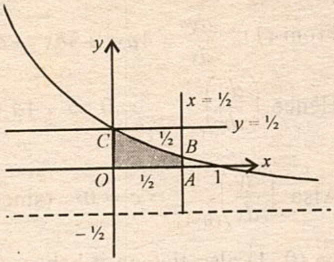
$\therefore$ Minimum odd value of a is 3.

3. $2^{|x|} \cdot |y| + 2^{|x|-1} \leq 1$ ...(1)

Clearly, this region is symmetrical about x and y axis

Let, $x \geq 0,\ y \geq 0$, equation (1) gives,

$2^x \cdot y + 2^{x-1} \leq 1$



$\Rightarrow y \leq \dfrac{1-2^{x-1}}{2^x}$

$= 2^{-x} - \dfrac{1}{2}$

Clearly, bounded region in the first quadrant is $OABC$. The required area is 4 times the area of the region $OABC$

$$\text{Required area} = 4\int_0^{1/2}\left(2^{-x} - \frac{1}{2}\right)dx = 4\left(-\frac{2^{-x}}{\ln 2} - \frac{x}{2}\right)_0^{1/2}$$

$$= \frac{4}{\ln 2}(1 - 2^{-1/2}) - 1$$

4. $f'(x) = \lim\limits_{h \to 0}\dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x)f^2(h/2) - f(x)}{h} = f(x)\lim_{h\to 0}\frac{f^2(h/2) - 1}{h}$$

$$= f(x)\lim_{h\to 0}\frac{(f(h/2) - 1)}{(2\times(h/2))}\cdot\left(f\left(\frac{h}{2}\right) + 1\right)$$

$$= f(x)f'(0) = f(x)\ln 2 \qquad [\text{since, } f(0) = 1]$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \ln(2) \Rightarrow f(x) = 2^x + c$$

Since, $f(0) = 1 \Rightarrow c = 0 \Rightarrow f(x) = 2^x$

$$f(x) + f(2x) + ... + f(nx) = 2^x + 2^{2x} + ...2^{nx}$$

$$= \frac{2^x(2^{nx} - 1)}{2^x - 1} = \frac{f(x)(f(nx) - 1)}{f(x) - 1}.$$

5. Given that $\dfrac{d}{dx}H(x) > 2cxH(x)$

$$\Rightarrow \frac{d}{dx}H(x) - 2cxH(x) > 0 \Rightarrow \frac{d}{dx}(H(x)e^{-cx^2}) > 0$$

$\Rightarrow H(x)e^{-cx^2}$ is an increasing function.

But $H(x_0) = 0$ and e^{-cx^2} is always positive

$\Rightarrow H(x) > 0$ for all $x > x_0$

$\Rightarrow H(x)$ cannot be zero for any $x > x_0$.

6. $\left[\dfrac{x-1}{2}\right] = 0$ for $0 \leq \dfrac{x-1}{2} < 1$ for $1 \leq x < 3$

$\Rightarrow \dfrac{5}{\left[\dfrac{x-1}{2}\right]}$ is defined for all $x \in R - [1, 3)$

$\sin^{-1}x^2$ is defined for $-1 \leq x \leq 1$

$\Rightarrow 3^{\sin^{-1}x^2}$ is defined for $x \in [-1, 1]$

$(7x + 1)!$ is defined for $7x + 1 \geq 0$ with

$7x + 1 \in N \cup \{0\} \Rightarrow x \in \left\{-\dfrac{1}{7},\ 0,\ \dfrac{1}{7},\ 1,\ \dfrac{2}{7},\ ...\right\}$

$\dfrac{1}{\sqrt{x+1}}$ is defined for $x \in (-1, \infty)$

taking intersection of all these domains

$D_1 = R - [1, 3)$

$D_2 = N \cup \{0\} = x \in \left\{-\dfrac{1}{7}, 0, \dfrac{1}{7}, \dfrac{2}{7}, ...\right\}$

$D_3 = (-1, \infty)$, Domain $= D_1 \cap D_2 \cap D_3 = \left\{-\dfrac{1}{7}\right\}$.

7. Put $y = \tan\theta$

$$\Rightarrow \int\frac{dy}{y^2(1+y^2)^3} = \int\frac{\cos^6\theta}{\sin^2\theta}d\theta = \int\frac{(1-\sin^2\theta)^3\,d\theta}{\sin^2\theta}$$

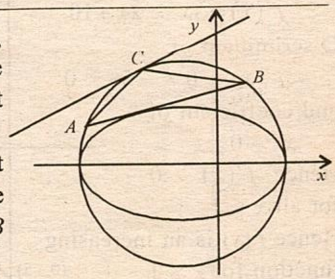
$$= \int\frac{(1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta)d\theta}{\sin^2\theta}$$

$$= \int(\operatorname{cosec}^2\theta - 3 + 3\sin^2\theta - \sin^4\theta)d\theta$$

$$= -\cot\theta - 3\theta + \frac{3}{2}\int(1 - \cos 2\theta)d\theta - \frac{1}{4}\int(1 - \cos 2\theta)^2\,d\theta$$

$$= -\frac{1}{y} - \frac{15}{8}\tan^{-1}y - \frac{1}{2}\sin(2\tan^{-1}y) - \frac{1}{32}\sin(4\tan^{-1}y) + c.$$

8. For a fixed line AB, area of ΔACB will be fixed only when tangent at C is parallel to AB. Let equations of tangent AB and tangent to the circle at C parallel to AB are $y = mx + \sqrt{4m^2 + 1}$



and $y = mx + 2\sqrt{1 + m^2}$ since, $y = mx + \sqrt{4m^2 + 1}$ cuts the circle $x^2 + y^2 = 4$ at A and B then :

$$x^2 + (mx + \sqrt{4m^2 + 1})^2 = 4$$

$$\Rightarrow x_1 + x_2 = \frac{-2m\sqrt{4m^2 + 1}}{1 + m^2} \text{ and } x_1x_2 = \frac{4m^2 - 3}{1 + m^2}$$

$$\Rightarrow (x_1 - x_2)^2 = \frac{4m^2(4m^2 + 1)}{(1 + m^2)^2} - \frac{4(4m^2 - 3)}{1 + m^2}$$

$$= \frac{16m^4 + 4m^2 - 16m^2 - 16m^4 + 12 + 12m^2}{(1+m^2)^2} = \frac{12}{(1+m^2)^2}$$

$$\Rightarrow (y_1 - y_2) = m(x_1 - x_2)$$

$$\Rightarrow AB = \sqrt{1+m^2} \cdot \frac{\sqrt{12}}{(1+m^2)} = \frac{\sqrt{12}}{\sqrt{1+m^2}}$$

$\Rightarrow$ Altitude of triangle ABC = Distance between tangents

$$= \left| \frac{2\sqrt{1+m^2} - \sqrt{4m^2+1}}{\sqrt{1+m^2}} \right|$$

$$\text{Area of } \Delta ABC = \frac{1}{2} \cdot \frac{\sqrt{12}}{\sqrt{1+m^2}} \cdot \frac{(2\sqrt{1+m^2} - \sqrt{4m^2+1})}{\sqrt{1+m^2}}$$

$$A = \sqrt{3}\left(\frac{2}{\sqrt{1+m^2}} - \frac{\sqrt{4m^2+1}}{1+m^2} \right)$$

$$\frac{dA}{dm} = \sqrt{3} \cdot \left(\frac{-m}{(1+m^2)^{3/2}} - \frac{2m - 4m^3}{(1+m^2)^2 \sqrt{4m^2+1}} \right)$$

$$= -\sqrt{3}m \left(\frac{\sqrt{1+m^2}\sqrt{4m^2+1} + 2 - 4m^2}{(1+m^2)^2 \sqrt{4m^2+1}} \right) = 0$$

$\Rightarrow m = 0$, At $m = 0$, $\dfrac{dA}{dm}$ changes sign from positive to negative. So there is maximum.

Maximum area $= \sqrt{3}$.

9. For $x \le 1$;

$$f'(x) = 3x^2 - 2x + 10$$

Discriminant of

$f'(x) = 0 = -56 < 0$

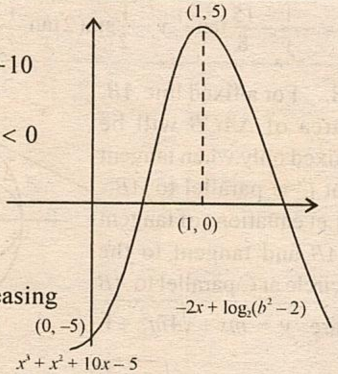
and coefficient of

$x^2 > 0$

Hence $f'(x) > 0$

for all $x \le 1$

Hence $f(x)$ is an increasing function for $x \le 1$



For $x > 1$; $f'(x) = -2$

$\Rightarrow f(x)$ is a decreasing function for $x > 1$

$f(x)$ will have the greatest value at $x = 1$ if

$$\lim_{x \to 1^+} f(x) \le f(1) \Rightarrow \lim_{h \to 0} f(1+h) \le f(1)$$

$$\Rightarrow -2 + \log_2(b^2 - 2) \le 5 \Rightarrow \log_2(b^2 - 2) \le 7$$

$$\Rightarrow b^2 - 2 \le 2^7 \Rightarrow b^2 \le 130$$

Again $b^2 - 2 > 0$ for $\log_2(b^2 - 2)$ to be defined

$\Rightarrow 2 < b^2 < 130$

$$\therefore b \in \left[-\sqrt{130}, -2\right) \cup \left(\sqrt{2}, \sqrt{130}\right]$$

10. $y = ax^4 + bx^3 + cx + d$ ...(1)

y touches x-axis at $(-1, 0)$

so $(-1, 0)$ lies on it and $dy/dx = 0$

so, $0 = a - b - c + d$ ...(2)

From (1), $\dfrac{dy}{dx} = 4ax^3 + 3bx^2 + c$ ...(3)

Hence $\left(\dfrac{dy}{dx}\right)_{(-1,\,0)} = 0 \Rightarrow -4a + 3b + c = 0$ ...(4)

Also $\left(\dfrac{dy}{dx}\right)_{(0,\,1)} = c = 0$ (since curve touches (0, 1))

$\Rightarrow (0, 1)$ also lies on it hence $d = 1$

Putting values of c and d in (2) and (4), solving for a and b we get $a = 3$, $b = 4$

Therefore (3) becomes $\dfrac{dy}{dx} = 12x^3 + 12x^2$

Now $\dfrac{dy}{dx} < 0 \Rightarrow 12x^3 + 12x^2 < 0$

$\Rightarrow 12x^2(x+1) < 0 \Rightarrow x < -1.$ ■